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Thermodynamic behavior of an area-preserving multibaker map with energy*

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Abstract. A multibaker map with "kinetic energy" is proposed which incorporates an external field. The map is volume-preserving, time-reversal symmetric and conserves total energy. In an appropriate macroscopic limit, the particle distribution is shown to obey a Smoluchowski-type equation. For the cases without any external field and with a constant external force, the nonequilibrium stationary states are constructed by solving the evolution equation of the partially integrated distribution functions. These states are described by singular functions such as incomplete Takagi functions and Lebesgue's singular functions. In an appropriate macroscopic limit, the mass flows for the stationary states are shown to be identical to the ones expected from the Smoluchowski equation and a "heat flow" proportional to the local energy gradient appears. The Gaspard–Gilbert–Dorfman entropy production is calculated for the stationary states and is shown to be positive. Particularly, for the case with a constant external force, when the energy distribution is independent of the spatial distribution, the entropy production reduces to the one consistent with classical thermodynamics. The result shows that there exists a volume-preserving driven multibaker map whose entropy production is consistent with classical thermodynamics.

Key words: Nonequilibrium stationary state – Flux boundary conditions – Singular measure – Multibaker map

1 Introduction

The emergence of irreversible behaviors and the understanding of microscopic entropy production are long-standing problems in statistical mechanics [1–5]. Recently, stimulated by the progress of dynamical systems theory, the problems have been extensively studied mainly from two different points of view (Ref. [6] and references therein). On the one hand [6–12], the dynamics is modified in such a way that a fictitious damping force is introduced to avoid an uncontrolled growth of the kinetic energy due to an external driving force, while it preserves time reversibility. For such systems, called thermostated systems, nonequilibrium stationary states are realized as Sinai–Ruelle–Bowen (SRB) states and they fully characterize transport properties such as a transport law, transport coefficients and their fluctuations. For example, Ohm's law and Einstein's relation have been rigorously proved for the driven thermostated Lorentz gas [8]. For thermostated systems, the Gibbs entropy of a nonequilibrium stationary state is not well defined as a result of the singular nature of the SRB states, but the Gibbs entropy production is well defined and is related to the thermodynamic entropy production (cf. Refs. [6, 10] and references therein). On the other hand [4, 13–18], nonequilibrium stationary states have been investigated for open Hamiltonian systems such as the Lorentz gas and an area-preserving multibaker map. In this case, the nonequilibrium stationary states are described by fractal distributions similar to the SRB states. The latter approach is based on the belief that the underlying microscopic dynamics is Hamiltonian and that the nonequilibrium states are established by boundary conditions. In this case, because of the fractality of the stationary distributions, the entropy production is calculated with the aid of the coarse grained entropy and, for an open dyadic multibaker map [18], it reduces to a thermodynamic expression in the macroscopic limit. The interrelation of the two approaches has been discussed in Refs. [19–21]. We note that, in both approaches, the Gallavotti–Cohen hypothesis [11] plays an important role. It asserts that the microscopic dynamics of an N-body system for large N is of hyperbolic character.

The multibaker map [13] is a lattice extension of the conventional baker transformation, which exhibits a deterministic diffusion. For the simplest dyadic area-

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preserving multibaker map, nonequilibrium stationary states were constructed with the aid of Takagi-type functions [16, 18]. The multibaker maps and their generalizations are extensively used to study transport properties including the problem of irreversible entropy production [12, 16, 18–21]. However, the multibaker maps used so far can only describe the mass transport since they do not have energy. Also, as pointed out by Breymann et al. [20], the conventional multibaker map mimicking diffusion with drift can have an entropy production consistent with classical thermodynamics, only when the dynamics is dissipative. Therefore, it is interesting to propose and investigate an area-preserving driven multibaker map which includes energy and whose entropy production is consistent with classical thermodynamics.

In this paper, we introduce and investigate a multibaker map with energy and which is volume-preserving and time-reversal symmetric. As shown in Sect. 2, an external field can be introduced by requiring the conservation of total energy and of the phase-space volume. The evolution equation of the measure is derived in Sect. 3 and the macroscopic limit is studied in Sect. 4. It is found that, in the macroscopic limit, the local energy distribution function obeys the Smoluchowski-type equation. In Sect. 5, microscopic distributions for the stationary states are investigated in detail for the cases without any external field and with a constant external force. In both cases, the nonequilibrium stationary states are described by fractal distributions and their transport properties agree with a phenomenological theory based on the Smoluchowski equation. The entropy production for the nonequilibrium stationary states is investigated in Sect. 6 and is shown to be consistent with classical thermodynamics when there is a drift in the mass transport. Section 7 is devoted to concluding remarks. In the Appendix, we derive the stationary distributions for the dissipative multibaker map introduced by Tél and coworkers [19, 20] and compare the results with the conservative multibaker map with energy.

2 Multibaker map with energy

We introduce a multibaker model which is a caricature of the Lorentz gas and which incorporates both energy and an applied field. First, we observe that the phase-space dimension is 3 for the first return map of the Lorentz gas: 2 for the velocity direction and the hitting angle at the scatterer and 1 for the kinetic energy. The conventional multibaker maps mimic the return map of the Lorentz gas on the constant kinetic energy surface. It is therefore natural to introduce a "kinetic energy variable" E in addition to the internal coordinates (x, y) for each cell n. We also note that the phase-space volume of the Lorentz gas at each scatterer is proportional to the square root of the kinetic energy E. Hence in our multibaker model, we assume that the sectional phase-space area for each energy value E depends on E. Then the phase space Γ of our model is (cf. Fig. 1)

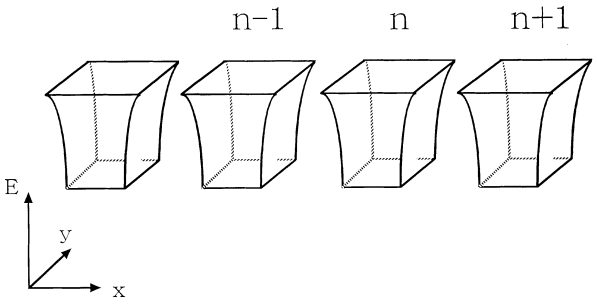

Fig. 1. Schematic representation of the phase space Γ. The sectional area at "kinetic energy" E depends on E

$$\Gamma = \{(n, x, y, E) | n \in \mathbf{Z}, \quad E \in \mathbf{R}^+, \\ 0 \le x \le a(E),\ 0 \le y \le a(E)\}\ , \tag{1}$$

where $\mathbf{Z}$ and $\mathbf{R}^+$ stand for the sets of integers and of positive real numbers, respectively, and $a(E)$ is a positive and increasing function of the kinetic energy E.

Now we turn to the dynamics, which is invertible, area-preserving, energy-conserving and time-reversal symmetric. In order to control the diffusion coefficient, we use a three-strip version. As in the Lorentz gas, the dynamics depends on the presence of an external field. So we consider the two cases separately.

2.1 Free dynamics

When there is no external field, the system exhibits pure diffusion and the kinetic energy is preserved. Thus, we have (cf. Fig. 2)

$$B_{\Phi=0}(n,x,y,E) = \begin{cases} \left(n-1, \frac{x}{l}, l\ y, E\right), & x \in [0, la(E)), \\ \left(n, \frac{x - la(E)}{s}, sy + la(E), E\right), & x \in [la(E), (1-l)a(E)), \\ \left(n+1, \frac{x-(1-l)a(E)}{l}, ly + (1-l)a(E), E\right), & x \in [(1-l)a(E), a(E)]\ , \end{cases} \tag{2}$$

where the parameter l $(0 < l \le 1/2)$ determines the diffusion coefficient, $s = 1 - 2l$, and the subscript $\Phi = 0$ stands for the absence of the external field. In general, the diffusion coefficient may depend on the kinetic energy, but for simplicity, it is assumed to be independent of E. The map $B_{\Phi=0}$ is clearly volume-preserving.

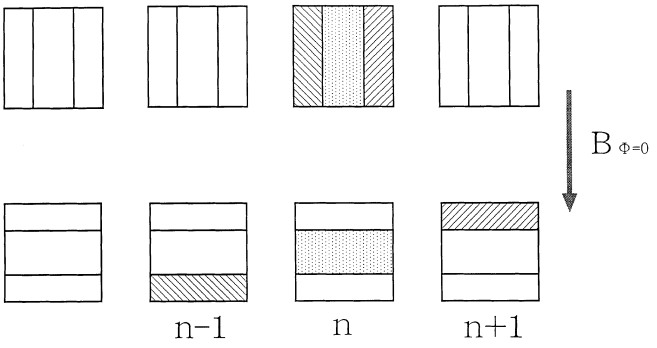

Fig. 2. The multibaker map on the constant total energy surface: the case without an external field

Moreover, it is time-reversal symmetric in the sense that it satisfies

$$I\ B_{\Phi=0}\ I = B_{\Phi=0}^{-1}\ , \tag{3}$$

where the involution I is given by

$$I\ (n,x,y,E) = (n, a(E) - y, a(E) - x, E)\ . \tag{4}$$

2.2 Dynamics under an external field

Now we consider the case where there exists an external field derived from a potential energy Φ which is a function of the site coordinate n. We require the dynamics to preserve both the total energy and the phase-space volume and to be invertible and time-reversal symmetric with respect to the involution I. These requirements determine the dynamics as follows.

1. As a result of the total energy conservation, when a point of the site n is mapped to the site $n \pm 1$, its kinetic energy changes from E to $E - \Delta_{\pm}\Phi(n)$, where $\Delta_{\pm}\Phi(n) = \Phi(n \pm 1) - \Phi(n)$ is the potential energy difference.

2. Because of the energy dependence of the phase-space volume, the transition rates $l_n^{\pm}$ from the site n to the sites $n \pm 1$ should depend on the external potential Φ as well as on the site coordinate n.

3. Since the map B_{Φ} is onto, conditions 1 and 2 give

$$l_{n-1}^{+} a[E - \Delta_{-}\Phi(n)]^2 + s_n a(E)^2 + l_{n+1}^{-} a[E - \Delta_{+}\Phi(n)]^2 = a(E)^2\ , \tag{5}$$

where s_n is the transition rate from the site n to itself: $s_n = 1 - l_n^{+} - l_n^{-}$. Differentiating Eq. (5) with respect to $\Phi(n+1)$ and setting $\Phi \equiv 0$, one obtains a differential equation for $a(E)$:

$$-2la(E)\ \frac{da(E)}{dE} + \left(\frac{\partial\left(l_{n-1}^{+} + s_n + l_{n+1}^{-}\right)}{\partial\Phi(n+1)}\right)_{\Phi=0} a(E)^2 = 0\ ,$$

which has a solution $a(E) = a\exp(cE)$ with constants $a(>0)$ and c. As $a(E)$ is an increasing function of E, $c > 0$ and one can always set $c = 1$ by changing the unit of the energy, i.e., $a(E) = a\exp(E)$.

4. The time-reversal symmetry with respect to the involution I imposes the condition

$$l_n^{-} = l_{n-1}^{+}\ \exp\{2[\Phi(n) - \Phi(n-1)]\}\ . \tag{6}$$

5. As the self-transition rate s_n is not fixed by the above prescriptions, we assume that it is independent of the potential Φ as Φ is constant over each cell, i.e., $s_n = 1 - 2l$.

In short, we obtain the following (cf. Fig. 3)

For $x \in [0, l_n^{-}a(E))$,

$$B_{\Phi}(n,x,y,E) = \left(n-1,\ \frac{x}{l_{n-1}^{+}\exp[-\Delta_{-}\Phi(n)]},\right.$$
$$\left. l_{n-1}^{+}\exp[-\Delta_{-}\Phi(n)]\ y,\ E - \Delta_{-}\Phi(n)\right)\ . \tag{7}$$

For $x \in [l_n^{-}a(E), (1 - l_n^{+})a(E))$,

$$B_{\Phi}(n,x,y,E) = \left(n,\ \frac{x - l_n^{-}a(E)}{s},\ sy + l_n^{+}a(E),\ E\right)\ . \tag{8}$$

For $x \in [(1 - l_n^{+})a(E), a(E)]$,

$$B_{\Phi}(n,x,y,E) = \left(n+1,\ \frac{x - (1 - l_n^{+})a(E)}{l_{n+1}^{-}\exp[-\Delta_{+}\Phi(n)]},\right.$$
$$\left[l_{n+1}^{-}y + (1 - l_{n+1}^{-})a(E)\right]\exp[-\Delta_{+}\Phi(n)],$$
$$\left. E - \Delta_{+}\Phi(n)\right)\ . \tag{9}$$

In the above, the parameters $l_n^{\pm}$ satisfy

$$l_n^{+} + l_n^{-} = 2l\ , \tag{10}$$

$$l_n^{-} = l_{n-1}^{+}\ \exp\{2[\Phi(n) - \Phi(n-1)]\}\ , \tag{11}$$

where $\Delta_{\pm}\Phi(n) = \Phi(n \pm 1) - \Phi(n)$ and $s = 1 - 2l$. The map B_{Φ} is invertible and time-reversal symmetric in the sense that $IB_{\Phi}I = B_{\Phi}^{-1}$ and preserves the total energy $E + \Phi$ and the phase-space volume. Also, for a fixed value of the total energy $E + \Phi$, the map is hyperbolic.

We note that the inverse of B_{Φ} is given by the following equations.

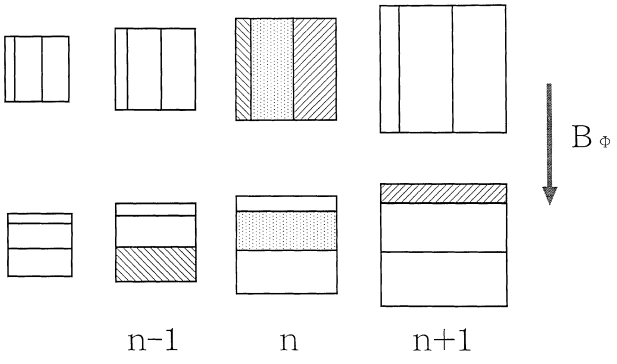

Fig. 3. The multibaker map on the constant total energy surface: the case with a constant external force

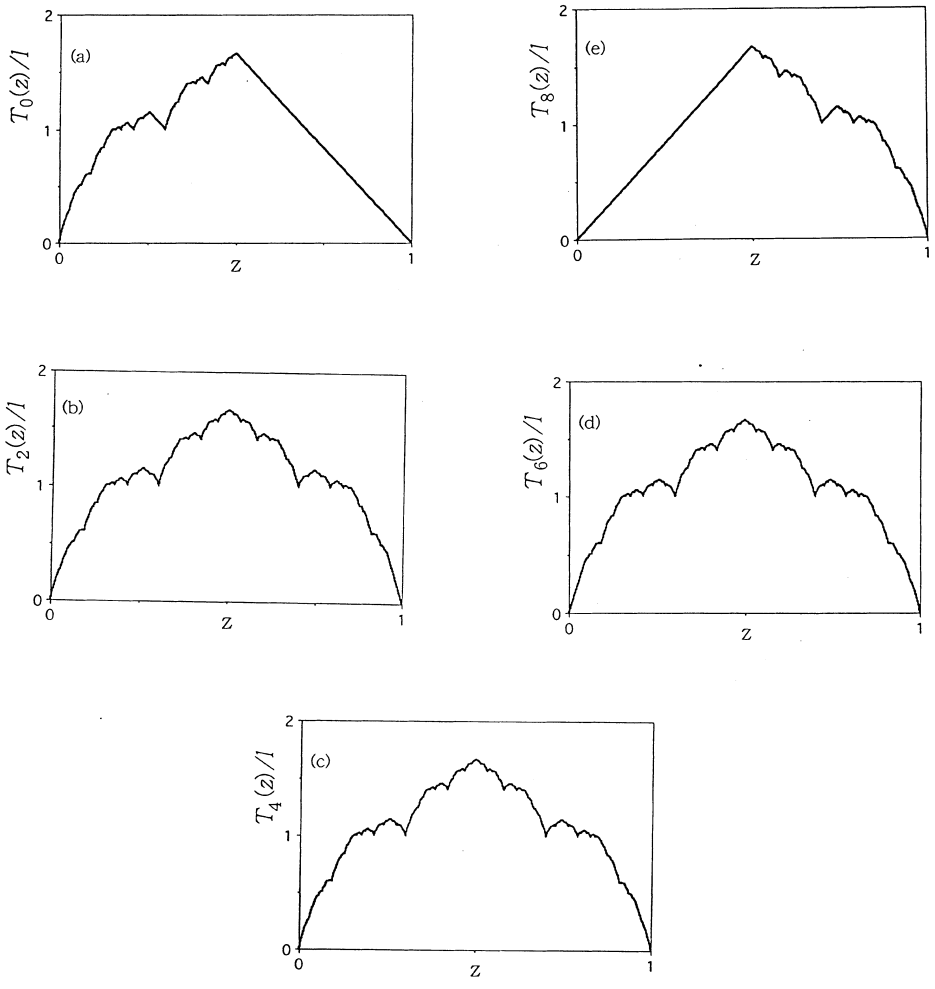

Fig. 4a–e. Incomplete Takagi functions $T_n(z)$ versus z associated with the intracell distributions along the contracting y-direction for the case without an external field. The ordinate is $T_n(z)/l$. The system consists of nine cells and n is the cell coordinate. The parameter l has the value $l = 0.3$

For $y \in [0, l_n^+ a(E))$,

$$B_\Phi^{-1}(n,x,y,E) = \left(n+1,\, l_{n+1}^- \exp[-\Delta_+\Phi(n)]\; x,\; \frac{y}{l_{n+1}^- \exp[-\Delta_+\Phi(n)]},\;\; E - \Delta_+\Phi(n)\right) . \tag{12}$$

For $y \in [l_n^+ a(E), (1-l_n^-)a(E))$,

$$B_\Phi^{-1}(n,x,y,E) = \left(n,\; sx + l_n^- a(E),\; \frac{y - l_n^+ a(E)}{s},\; E\right) . \tag{13}$$

For $y \in [(1-l_n^-)a(E), a(E)]$,

$$B_\Phi^{-1}(n,x,y,E) = \left(n-1,\;\; \left[l_{n-1}^+\; x + (1-l_{n-1}^+)a(E)\right] \times \exp[-\Delta_-\Phi(n)],\; \frac{y-(1-l_n^-)a(E)}{l_{n-1}^+ \exp[-\Delta_-\Phi(n)]},\; E - \Delta_-\Phi(n)\right) . \tag{14}$$

3 Evolution equation and flow

We consider the equation of motion of the states, which are represented by measures. Since B_Φ is volume-preserving, we consider the time evolution equation of the partially integrated distribution function G_t for a fixed total energy E:

$$G_t(n,x,y,E) \equiv \int_0^y dy' \rho_0\left(B_\Phi^{-t}[n,x,y',E-\Phi(n)]\right) , \tag{15}$$

where ρ_0 is the initial distribution function and E denotes here the total energy. Its evolution equation can be obtained from Eqs.(12)–(14).

For $y \in [0, l_n^+ a_n(E))$,

$$G_{t+1}(n,x,y,E) = l_{n+1}^- \exp[-\Delta_+\Phi(n)] \times G_t\left(n+1,\, l_{n+1}^- \exp[-\Delta_+\Phi(n)]\; x,\; \frac{y}{l_{n+1}^- \exp[-\Delta_+\Phi(n)]},\;\; E\right) . \tag{16}$$

For $y \in [l_n^+ a_n(E), (1-l_n^-)a_n(E)]$,

$$G_{t+1}(n,x,y,E) = l_{n+1}^- \exp[-\Delta_+\Phi(n)] \times G_t\left(n+1, l_{n+1}^- \exp[-\Delta_+\Phi(n)]\; x,\; a_{n+1}(E),\;\; E\right) + s\; G_t\left(n,\; sx + l_n^- a_n(E),\; \frac{y - l_n^+ a_n(E)}{s},\; E\right) . \tag{17}$$

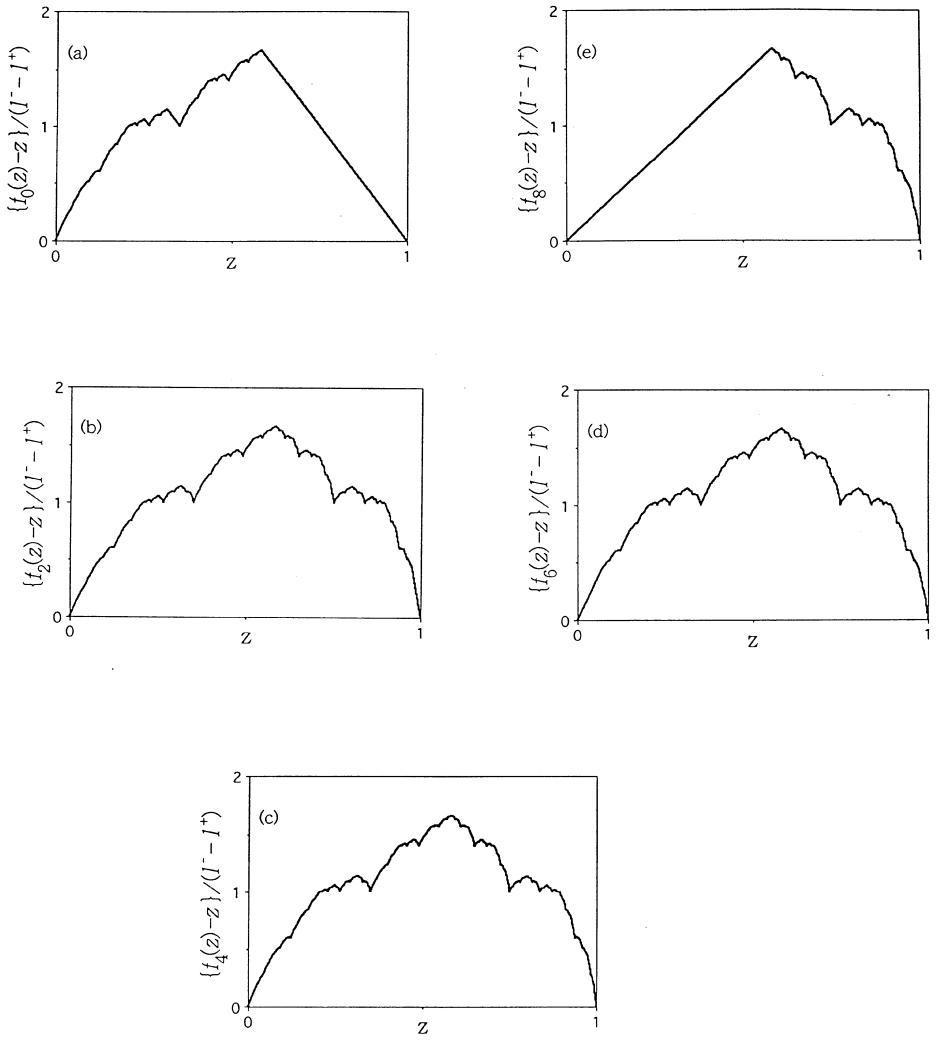

Fig. 5a–e. Incomplete Lebesgue singular functions $f_n(z)$ versus z associated with the intracell distributions along the contracting y-direction for the case with a constant external force. In order to compare the distributions to those without an external field, the deviation from the uniform distribution defined by $[f_n(z) - z]/(l^- - l^+)$ is plotted. The system consists of nine cells and n is the cell coordinate. The parameters are $l^+ = 0.35$ and $l^- = 0.25$

For $y \in [(1 - l_n^-)a_n(E), a_n(E)]$,

$$
\begin{aligned}
G_{t+1}(n,x,y,E) &= l_{n+1}^- \exp[-\Delta_+\Phi(n)] \\
&\quad \times G_t\big(n+1, l_{n+1}^- \exp[-\Delta_+\Phi(n)]\,x,\ a_{n+1}(E),\ E\big) \\
&\quad + s\,G_t\big[n,\ sx + l_n^- a_n(E), a_n(E),\ E\big] \\
&\quad + l_{n-1}^+ \exp[-\Delta_-\Phi(n)] \\
&\quad \times G_t\Bigg(n-1,\ \big[l_{n-1}^+ x + (1 - l_{n-1}^+)a_n(E)\big] \\
&\quad \times \exp[-\Delta_-\Phi(n)], \frac{y - (1 - l_n^-)a_n(E)}{l_{n-1}^+ \exp[-\Delta_-\Phi(n)]}, E\Bigg)\ , \qquad (18)
\end{aligned}
$$

where $a_n(E) \equiv a[E - \Phi(n)] = a\exp[E - \Phi(n)]$.

Now we turn to the flow. From Eq. (18), the probability distribution $\Pi_t(n,E) \equiv \int_0^{a_n(E)} dx G_t[n,x,a_n(E),E]$ per site and per energy is found to obey an equation of continuity

$$
\begin{aligned}
&\Pi_{t+1}(n,E) - \Pi_t(n,E) \\
&\qquad = -J_{n|n+1}(E,t) + J_{n-1|n}(E,t)\ , \qquad (19)
\end{aligned}
$$

where the probability flow $J_{n|n+1}(E,t)$ from site n to site $n+1$ at time t is given by

$$
\begin{aligned}
J_{n|n+1}(E,t) &\equiv \int_{(1-l_n^+)a_n(E)}^{a_n(E)} \mathrm{d}x G_t[n,x,a_n(E),E] \\
&\quad - \int_0^{l_{n+1}^- a_{n+1}(E)} \mathrm{d}x G_t[n+1,x,a_{n+1}(E),E]\ . \qquad (20)
\end{aligned}
$$

As easily seen, the evolution Eqs. (16)–(18) give a stationary solution

$$
G(n,x,y,E) = \rho_{\mathrm{eq}}(E)y\ . \qquad (21)
$$

For this state, the distribution $\Pi(n,E)$ per site and per energy is given by

$$
\Pi(n,E) = a^2 \mathrm{e}^{2E} \rho_{\mathrm{eq}}(E) \exp[-2\Phi(n)]\ , \qquad (22)
$$

and the probability flow vanishes

$$
\begin{aligned}
J_{n|n+1}(E) &= l_n^+ \Pi(n,E) - l_{n+1}^- \Pi(n+1,E) \\
&= \left\{1 - \frac{l_{n+1}^-}{l_n^+} \exp[-2\Delta_+\Phi(n)]\right\} l_n^+ \Pi(n,E) = 0\ . \qquad (23)
\end{aligned}
$$

This stationary state corresponds to the equilibrium state under the potential Φ.

4 Macroscopic limit

In this section, we study the macroscopic limit of the asymptotic evolution of the state assuming that the initial density $\rho_0[n,x,y,E-\Phi(n)]$ is continuously differentiable with respect to x. Then from Eqs. (16)–(18), we find that $G_t(n,x,y,E)$ is continuously differentiable with respect to x for all $t>0$ and its derivative obeys the inequalities

$$\|\partial_x G_t\| \le \lambda \|\partial_x G_{t-1}\| \le \cdots \le \lambda^{t-1}\|\partial_x G_1\| \ , \tag{24}$$

where $\|\partial_x G_t\| \equiv \sup_{n,x,y,E}|\partial_x G_t(n,x,y,E)|$ and

$$\lambda \equiv \sup\{l_{n+1}^{-2}\exp[-2\Delta_+\Phi(n)] + (1-2l)^2 + l_{n-1}^{+2}\exp[-2\Delta_-\Phi(n)]\} \le \sup(l_n^{\pm}, 1-2l) < 1 \ .$$

In the above, we have used $l_{n\pm1}^{\mp}\exp[-2\Delta_\pm\Phi(n)] = l_n^{\pm}$ and $l_n^+ + l_n^- = 2l$. Therefore, for long time $\lambda^t \ll 1$, G_t becomes x-independent. In this asymptotic regime, we study the macroscopic limit of the evolution equation for the probability distribution $\Pi_t(n,E)$ per site and per energy. By setting $y = a_n(E)$, dropping the x-dependence in Eq. (18) and multiplying the result by $a_n(E)$, one finds

$$\Pi_{t+1}(n,E) - \Pi_t(n,E) = l_{n+1}^-\Pi_t(n+1,E) + l_{n-1}^+\Pi_t(n-1,E) - 2l\Pi_t(n,E) \ . \tag{25}$$

The macroscopic limit is established by scaling l as

$$l = \frac{\tau}{d^2}D \ , \tag{26}$$

and letting $\tau \to 0$ and $d \to 0$ with finite D, where τ and d are the unit time step and site spacing, respectively [20]. The finite quantity D is the diffusion coefficient. Correspondingly, the potential Φ and the probability density per energy $\pi \equiv \Pi/d$ are smooth functions of $X \equiv nd$ and $T \equiv m\tau$. From Eq. (11), one finds

$$l_n^{\pm} = l \mp dl\frac{\partial\Phi}{\partial X} + O(d^2) \ .$$

Therefore, Eq. (25) reduces to

$$\frac{\pi_{T+\tau}(X,E) - \pi_T(X,E)}{\tau} = l\frac{d^2}{\tau}\left[\frac{\partial}{\partial X}\left(2\frac{\partial\Phi(X)}{\partial X}\pi_T(X,E) + \frac{\partial\pi_T(X,E)}{\partial X}\right)\right] + O(d)$$

or by taking the limit $d \to 0$ and $\tau \to 0$,

$$\frac{\partial\pi_T(X,E)}{\partial T} = D\frac{\partial}{\partial X}\left(2\frac{\partial\Phi(X)}{\partial X}\pi_T(X,E) + \frac{\partial\pi_T(X,E)}{\partial X}\right) \ , \tag{27}$$

which is a Smoluchowski-type equation (e.g., Ref. [22]).

Similarly, the flow $J_{n|n+1}(E,t)$ is given by

$$J_{n|n+1}(E,t) = l_n^+\Pi_t(n,E) - l_{n+1}^-\Pi_t(n+1,E) \ , \tag{28}$$

which, in the macroscopic limit, reduces to

$$j(X,T;E) \equiv \frac{J_{n|n+1}(E,t)}{\tau} = -2D\frac{\partial\Phi(X)}{\partial X}\pi_T(X,E) - D\frac{\partial\pi_T(X,E)}{\partial X} \ . \tag{29}$$

Because of the positivity of the kinetic energy, the range of the total energy depends on the site coordinate X. Hence, the implications of Eqs.(27) and (29) should be discussed for each concrete case.

It is remarkable that, starting from an area-preserving and energy-conserving system under an external field, the Smoluchowski-type equation is derived without the use of a thermostat. The essential ingredient of this behavior seems to be the rapid increase of the phase space volume ($\sim e^{2E}$) as a function of the kinetic energy, which seems to prevent the uncontrolled growth of the kinetic energy due to the external force. At first sight, this assumption looks rather unphysical since, in the case of the Lorentz model, the phase-space volume increases gradually with kinetic energy ($\sim\sqrt{E}$). However, when the number of degrees of freedom increases, the phase-space volume grows more rapidly as a function of kinetic energy ($\sim E^{Nd/2}$, for a d-dimensional system consisting of N particles). Therefore, one can regard the exponential growth of the phase volume as effectively taking into account the properties of large systems.

5 Nonequilibrium stationary states

So far, we have discussed the macroscopic aspects of the general model. Since the detailed structures of the nonequilibrium stationary states as well as the relaxation modes depend on the explicit form of the external potential Φ, we study the nonequilibrium stationary states under a constant external force F, i.e., $\Phi(n) = Fn$ for a flux boundary condition, where the multibaker chain of length $N+1$ is connected to two particle reservoirs at both ends. The distributions of the reservoirs are assumed to be uniform with respect to the Lebesgue measure.

First we note that, since the system is uniform, $l_n^{\pm}$ is expected to be independent of the site coordinate n and thus,

$$l^+ + l^- = 2l \ , \quad l^- = e^{2F}l^+ \ ,$$

which gives

$$l^{\pm} = \frac{2l}{1+e^{\pm 2F}} \ . \tag{30}$$

Secondly, as shown in Sect. 4, the stationary state distribution is independent of the x-coordinate and thus, the cumulative distribution obeys the following equations.

For $y \in [0, l^+a_n(E))$,

$$G_\infty(n,y,E) = l^-e^{-F}G_\infty\left(n+1, \frac{y}{l^-e^{-F}}, E\right) \ . \tag{31}$$

For $y \in [l^+ a_n(E), (1-l^-)a_n(E))$,

$$G_\infty(n,y,E) = l^- \mathrm{e}^{-F} G_\infty[n+1,\ a_{n+1}(E),\ \ E] + s\ G_\infty\left(n,\ \frac{y - l^+ a_n(E)}{s},\ E\right) \ . \tag{32}$$

For $y \in [(1-l^-)a_n(E), a_n(E)]$,

$$G_\infty(n,y,E) = l^- \mathrm{e}^{-F} G_\infty[n+1,\ a_{n+1}(E),\ \ E] + s\ G_\infty[n,\ a_n(E),\ E] + l^+ \mathrm{e}^{F} G_\infty\left(n-1,\ \frac{y-(1-l^-)a_n(E)}{l^+ \mathrm{e}^F},\ E\right) \ , \tag{33}$$

where $a_n(E) \equiv a(E - nF) = a\exp(E - nF)$ and $n = 0, 1, \ldots N$. The flux boundary condition is imposed by requiring

$$G_\infty(-1,y,E) \ = \ \rho_-(E) y \ , \qquad G_\infty(N+1,y,E) \ = \ \rho_+(E) y \ . \tag{34}$$

By substituting $y = a_n(E)$ into Eq. (33) and noting $s = 1 - 2l$, one finds

$$2l G_\infty[n, a_n(E), E] = l^- \mathrm{e}^{-F} G_\infty[n+1, a_{n+1}(E), E] + l^+ \mathrm{e}^F G_\infty[n-1, a_{n-1}(E), E] \ . \tag{35}$$

Since the solution of Eq. (35) for $F = 0$ is qualitatively different from that for $F \neq 0$, we discuss them separately.

5.1 Free transport

When $F = 0$, Eq. (35) together with the boundary condition Eq. (34) gives the solution

$$G_\infty[n, a(E), E] \ = \frac{\rho_+(E) - \rho_-(E)}{N+2} a(E)(n+1) + \rho_-(E) a(E) \ . \tag{36}$$

Note that when $F = 0$, $a_n(E)$ is independent of the site coordinate n.

Then, the solution of Eqs. (31)–(33) is given by

$$G_\infty(n,y,E) = \frac{\rho_+(E) - \rho_-(E)}{N+2} \left[(n+1)y + a(E)\ T_n\left(\frac{y}{a(E)}\right)\right] + \rho_-(E) y \ , \tag{37}$$

where T_n is the incomplete Takagi-type function and is defined as the unique solution of the functional equation

$$T_n(z) = \begin{cases} l\ T_{n+1}(z/l) + z \ , & 0 \le z \le l \ , \\ s\ T_n[(z-l)/s] + l \ , & l \le z \le 1-l \ , \\ l\ T_{n-1}[(z-1+l)/l] + 1 - z \ , & 1-l \le z \le 1 \ , \end{cases} \tag{38}$$

with the boundary condition $T_{-1}(z) = T_{N+1}(z) = 0$. We remark that the distribution Eq. (37) is absolutely continuous with respect to the Lebesgue measure and becomes singular only for an infinitely long multibaker chain: $N \to \infty$.

Now we discuss the transport properties of the stationary state. Let P_n and $\bar{E}_n$ be, the particle distribution and the average total energy per site, respectively:

$$P_n \equiv \int_0^\infty dE\, a(E) G_\infty[n, a(E), E] \ , \qquad \bar{E}_n \equiv \int_0^\infty dE\ E a(E) G_\infty[n, a(E), E] \ .$$

Then

$$P_n = \frac{P_+ - P_-}{N+2}(n+1) + P_- \ , \tag{39}$$

$$\bar{E}_n = \frac{\bar{E}_+ - \bar{E}_-}{N+2}(n+1) + \bar{E}_- \ , \tag{40}$$

where

$$P_\pm \equiv \int_0^\infty dE\, a(E)^2 \rho_\pm(E) \ , \qquad \bar{E}_\pm \equiv \int_0^\infty dE\, E a(E)^2 \rho_\pm(E) \ .$$

Eqs. (39) and (40) imply that the particle distribution and the average total energy linearly depend on the site coordinate n.

For the stationary state Eq. (37), the particle flow $J^{\mathrm{M}}_{n|n+1} \equiv \int_0^\infty dE J_{n|n+1}(E)$ and the energy flow $J^{\mathrm{E}}_{n|n+1} \equiv \int_0^\infty dE\ E\ J_{n|n+1}(E)$ are given by

$$J^{\mathrm{M}}_{n|n+1} = -\ l \frac{P_+ - P_-}{N+2} = -l(P_{n+1} - P_n) \ , \tag{41}$$

$$J^{\mathrm{E}}_{n|n+1} = -\ l \frac{\bar{E}_+ - \bar{E}_-}{N+2} = -l(\bar{E}_{n+1} - \bar{E}_n) \ , \tag{42}$$

which show that the flows are proportional to the gradient of the distributions. Equation (41) is nothing but Fick's law of mass transport. These results imply that the system allows transport governed by a diffusion equation. The equality of the mass diffusion coefficient and the energy diffusion coefficient is simply due to the fact that, in the present system, transport is essentially controlled by the mass transport. We note that similar energy transport has been numerically studied for the Lorentz channel by Alonso et al. [23].

Finally, we briefly discuss the macroscopic limits of the flows. Let $p(X) \equiv P_n/d$ be the particle density and $\epsilon(X) \equiv \bar{E}_n/P_n$ be the average energy per particle. Then the particle flow and the energy flow can be rewritten as

$$j_{\mathrm{M}}(X) \equiv \frac{J^{\mathrm{M}}_{n|n+1}}{\tau} = -D \frac{\partial p(X)}{\partial X} \ , \tag{43}$$

$$j_{\mathrm{E}}(X) \equiv \frac{J^{\mathrm{E}}_{n|n+1}}{\tau} = -D\frac{\partial p(X)\epsilon(X)}{\partial X} = \epsilon(X) j_{\mathrm{M}}(X) - Dp(X)\frac{\partial \epsilon(X)}{\partial X} \quad , \tag{44}$$

which implies the appearence of the heat flow $j_{\mathrm{q}}(X)$ proportional to the gradient of the energy per particle [24]:

$$j_{\mathrm{q}}(X) = -Dp(X)\frac{\partial \epsilon(X)}{\partial X} \quad .$$

The details of the connection with linear nonequilibrium thermodynamics will be discussed elsewhere.

5.2 Transport under constant external force

When $F \neq 0$, Eq. (35) together with the boundary condition Eq. (34) gives

$$G_\infty[n, a_n(E), E] = A(E)\mathrm{e}^{-nF} + B(E)\mathrm{e}^{nF} \quad , \tag{45}$$

where the coefficients $A(E)$ and $B(E)$ are functions of $\rho_\pm(E)$:

$$A(E) = \frac{\exp[(N+2)F]\rho_-(E) - \exp[-(N+2)F]\rho_+(E)}{2\sinh(N+2)F} a(E) \quad , \tag{46}$$

$$B(E) = \frac{\rho_+(E) - \rho_-(E)}{2\sinh(N+2)F} a(E)\mathrm{e}^{-NF} \quad . \tag{47}$$

Then, the solution of Eqs.(31)–(33) is given by

$$G_\infty(n, y, E) = A(E)\frac{y}{a(E)} + B(E)\mathrm{e}^{nF} f_n\left(\frac{y}{a_n(E)}\right) \quad , \tag{48}$$

where f_n is a function similar to the Lebesgue singular function and is defined as the unique solution of the functional equation

$$f_n(z) = \begin{cases} l^- f_{n+1}(z/l^+), & 0 \le z \le l^+, \\ s\, f_n[(z-l^+)/s] + l^-, & l^+ \le z \le 1-l^-, \\ l^+ f_{n-1}[(z-1+l^-)/l^-] + 1 - l^+, & 1-l^- \le z \le 1 \quad , \end{cases} \tag{49}$$

with the boundary condition $f_{-1}(z) = f_{N+1}(z) = z$. In the case of a finite chain, the distribution Eq. (48) is absolutely continuous with respect to the Lebesgue measure and becomes singular only for an infinitely long multibaker chain: $N \to \infty$. Furthermore, as discussed in the Appendix, the stationary distribution Eq. (48) is identical, except for trivial scaling factors, to that for a dissipative multibaker map of Ref. [20] with "appropriate" dissipation.

Now we turn to the transport properties. In order to average with respect to energy distribution, we need one assumption: since the kinetic energy is positive, the domain of the function $\rho_-(E)$ is $E \ge -F$ and that of $\rho_+(E)$ is $E \ge (N+1)F$. However, we assume that the two functions $\rho_\pm(E)$ are nonzero in a common domain $E \ge \max[-F, (N+1)F] (\equiv E_0)$. Then by multiplying Eq. (45) by $a_n(E)$ and averaging it with respect to E over the interval $E \ge E_0$, one obtains the following particle distribution P_n

$$P_n = \bar{A}\mathrm{e}^{-2nF} + \bar{B} \quad , \tag{50}$$

where $\bar{A}$ and $\bar{B}$ are given by:

$$\bar{A} \equiv \int_{E_0}^{\infty} dE\, a(E)A(E) = \frac{P_- - P_+}{2\mathrm{e}^{-NF}\sinh(N+2)F} \quad , \tag{51}$$

$$\bar{B} \equiv \int_{E_0}^{\infty} dE\, a(E)B(E) = \frac{P_+ \exp[(N+2)F] - P_- \exp[-(N+2)F]}{2\sinh(N+2)F} \quad , \tag{52}$$

with $P_\pm$ the particle numbers per site at the edges:

$$P_- \equiv \mathrm{e}^{2F}\int_{E_0}^{\infty} dE\, a(E)^2 \rho_-(E) \quad ,$$

$$P_+ \equiv \exp[-2(N+1)F]\int_{E_0}^{\infty} dE\, a(E)^2 \rho_+(E) \quad .$$

Similarly, the average total energy per site E_n is given by

$$\bar{E}_n = \bar{A}_{\mathrm{E}}\mathrm{e}^{-2nF} + \bar{B}_{\mathrm{E}} \quad , \tag{53}$$

where $\bar{A}_{\mathrm{E}}$ and $\bar{B}_{\mathrm{E}}$ are

$$\bar{A}_{\mathrm{E}} \equiv \int_{E_0}^{\infty} dE\; E\; a(E)A(E) = \frac{\bar{E}_- - \bar{E}_+}{2\mathrm{e}^{-NF}\sinh(N+2)F} \quad , \tag{54}$$

$$\bar{B}_{\mathrm{E}} \equiv \int_{E_0}^{\infty} dE\; E\; a(E)B(E) = \frac{\bar{E}_+ \exp[(N+2)F] - \bar{E}_- \exp[-(N+2)F]}{2\sinh(N+2)F} \quad , \tag{55}$$

with $\bar{E}_\pm$ the average energies per site at the edges:

$$\bar{E}_- \equiv \mathrm{e}^{2F}\int_{E_0}^{\infty} dE\; E\; a(E)^2 \rho_-(E) \quad ,$$

$$\bar{E}_+ \equiv \exp[-2(N+1)F]\int_{E_0}^{\infty} dE\; E\; a(E)^2 \rho_+(E) \quad .$$

The particle flow $J^{\mathrm{M}}_{n|n+1}$ and total energy flow $J^{\mathrm{E}}_{n|n+1}$ are calculated as before

$$J^{\mathrm{M}}_{n|n+1} = -2l\frac{\mathrm{e}^F - \mathrm{e}^{-F}}{\mathrm{e}^F + \mathrm{e}^{-F}}\,\bar{B} \quad , \tag{56}$$

$$J^{\mathrm{E}}_{n|n+1} = -2l\frac{\mathrm{e}^F - \mathrm{e}^{-F}}{\mathrm{e}^F + \mathrm{e}^{-F}}\,\bar{B}_{\mathrm{E}} \quad . \tag{57}$$

All these results are expected from the Smoluchowski equation for constant external force. Hence, the system exhibits transport governed by a Smoluchowski equation.

This can also be seen in the macroscopic limit of the particle flow:

$$j_{\mathrm{M}}(X) = \frac{J^{\mathrm{M}}_{n|n+1}}{\tau} = -2D\mathscr{F}p(X) - D\frac{\partial p(X)}{\partial X} \quad , \tag{58}$$

where $\mathscr{F} \equiv F/d$ is the macroscopic field strength. The flow Eq. (58) is nothing but the flow for the Smoluchowski equation under constant external force $\mathscr{F}$. The macroscopic limit of the energy flow can be calculated as before

$$j_{\mathrm{E}}(X) = \frac{J^{\mathrm{E}}_{n|n+1}}{\tau} = \epsilon(X) j_{\mathrm{M}}(X) - Dp(X)\frac{\partial \epsilon(X)}{\partial X} \quad , \tag{59}$$

where $\epsilon(X)$ is the average energy per particle, and which again implies the existence of the heat flow $j_{\mathrm{q}}(X)$ proportional to the gradient of the energy per particle [24]:

$$j_{\mathrm{q}}(X) = -Dp(X)\frac{\partial \epsilon(X)}{\partial X} \quad .$$

6 Entropy production for stationary states

For thermostated systems, the temporal variation of the conventional Gibbs entropy provides a good measure of the irreversible entropy production because the dynamics is not conservative (Refs. [6, 10] and references therein). On the other hand, for conservative systems, the Gibbs entropy is kept constant with time and thus, it does not provide physically relevant information. As emphasized by Gaspard [18], for nonequilibrium stationary states represented by fractal distributions, the Gibbs entropy does not exist for infinitely large systems because of the fractality of the stationary states and the coarse grained entropy should be used as the microscopic entropy. Based on this view, Gaspard studied the entropy balance for deterministic systems, particularly for the dyadic multibaker map [18]. His approach was generalized by Gilbert and Dorfman [21] for systems with generating partitions. Similar approaches based on the coarse grained entropy balance have been developed by Tél and coworkers [19, 20] and by Nicolis and Daems [25]. In this section, following Gaspard's approach, we calculate the coarse grained entropy production for the stationary states studied in Sect. 5.

We begin with the coarse grained entropy of a set A with respect to the measure ν used by Gilbert and Dorfman [21]:

$$S(A : \{B_j\}) \equiv \sum_{B_j \subset A} \nu(B_j) \ln \frac{\mu_0(B_j)}{\nu(B_j)} \quad , \tag{60}$$

where $\{B_j\}$ is a partition of the phase space, μ_0 is the reference Lebesgue measure and the summation is taken over all B_j included in a set A. In Ref. [21], the authors assumed that the partition $\{B_j\}$ is generating, but here we do not assume it and rather follow the line of thoughts expressed in Ref. [18].

Now we briefly review the calculation of the entropy production. Let $S_t(A : \{B_j\})$ be the coarse grained entropy of set A with respect to the measure ν_t at time t, then the entropy change is given by

$$\Delta S_t(A : \{B_j\}) = S_{t+1}(A : \{B_j\}) - S_t(A : \{B_j\}) \quad ,$$

which is the sum of the entropy flow $\Delta_{\mathrm{e}} S_t(A : \{B_j\})$ and the entropy production $\Delta_i S_t(A : \{B_j\})$. Since the entropy flow is given by [18, 21]

$$\Delta_{\mathrm{e}} S_t(A : \{B_j\}) = S_t(B_\Phi^{-1}A : \{B_j\}) - S_t(A : \{B_j\}) \quad ,$$

one obtains

$$\Delta_i S_t(A : \{B_j\}) = S_{t+1}(A : \{B_j\}) - S_t(B_\Phi^{-1}A : \{B_j\}) \quad . \tag{61}$$

Since the map B_Φ preserves the Lebesgue measure, one has $\mu_0(B_j) = \mu_0(B_\Phi^{-1}B_j)$. Also $B_j \subset A$ implies $B_\Phi^{-1}B_j \subset B_\Phi^{-1}A$, and $\nu_{t+1}(B_j) = \nu_t(B_\Phi^{-1}B_j)$ by definition. Therefore,

$$\begin{aligned} S_{t+1}(A : \{B_j\}) &= \sum_{B_j \subset A} \nu_{t+1}(B_j) \ln \frac{\mu_0(B_j)}{\nu_{t+1}(B_j)} \\ &= \sum_{B_\Phi^{-1}B_j \subset B_\Phi^{-1}A} \nu_t(B_\Phi^{-1}B_j) \ln \frac{\mu_0(B_\Phi^{-1}B_j)}{\nu_t(B_\Phi^{-1}B_j)} \\ &= S_t(B_\Phi^{-1}A : \{B_\Phi^{-1}B_j\}) \quad , \end{aligned} \tag{62}$$

and hence,

$$\begin{aligned} &\Delta_i S_t(A : \{B_j\}) \\ &\quad = S_t(B_\Phi^{-1}A : \{B_\Phi^{-1}B_j\}) - S_t(B_\Phi^{-1}A : \{B_j\}) \quad . \end{aligned} \tag{63}$$

We note that an entropy production of a set A can be calculated with the aid of Eq. (63) only when the set $B_\Phi^{-1}A$ has the form

$$B_\Phi^{-1}A = \cup_{\text{some } i} B_\Phi^{-1}B_i = \cup_{\text{some } j} B_j \quad .$$

For open finite systems, the entropy production thus defined takes a finite value and remains constant when the size of the partitioning sets B_j is not too small and not too large and it vanishes when the size of B_j is vanishingly small [18, 21]. Based on this observation, we use a rough partition consisting of the cells

$$\begin{cases} \{(n,x,y,E) | 0 \le x \le a_n(E'), 0 \le y \le l^+ a_n(E'), E' \le E \le E' + \Delta E\}, \\ \{(n,x,y,E) | 0 \le x \le a_n(E'), l^+ a_n(E') \le y \le (1-l^-)a_n(E'), E' \le E \le E' + \Delta E\}, \\ \{(n,x,y,E) | 0 \le x \le a_n(E'), (1-l^-)a_n(E') \le y \le a_n(E'), E' \le E \le E' + \Delta E\} \quad , \end{cases} \tag{64}$$

and focus our attention on the macroscopic expression of the entropy production for a set $A = B_\Phi\{(n,x,y,E)|\ 0 \le x \le a_n(E), 0 \le y \le a_n(E), E_0 \le E\}$. Then, by a straightforward calculation and after taking the limit $\Delta E \to 0$, one finds

$$\Delta_i S(A:\{B_j\}) = \int_{E_0}^{\infty} dE\, s_i(E)\ , \tag{65}$$

where the entropy production per energy $s_i(E)$ is given by

$$\begin{aligned} s_i(E) = &- l^- \Pi_{n+1}(E) \ln \frac{e^{-2F}\Pi_n(E)}{\Pi_{n+1}(E)} \\ &- l^+ \Pi_{n-1}(E) \ln \frac{e^{2F}\Pi_n(E)}{\Pi_{n-1}(E)}\ , \end{aligned} \tag{66}$$

with $\Pi_n(E) \equiv a_n(E) G_\infty[n, a_n(E), E]$ the distribution per site and per energy. We recall that $\Pi_n(E)$ obeys the equation

$$2l\Pi_n(E) = l^- \Pi_{n+1}(E) + l^+ \Pi_{n-1}(E)\ , \tag{67}$$

$$l^\pm = \frac{2l}{1 + e^{\pm 2F}}\ . \tag{68}$$

Since $-\ln z$ is a concave function of z, the entropy production per energy $s_i(E)$ is non-negative

$$\begin{aligned} s_i(E) \ge &- \{l^- \Pi_{n+1}(E) + l^+ \Pi_{n-1}(E)\} \\ &\times \ln \frac{l^- e^{-2F}\Pi_n(E) + l^+ e^{2F}\Pi_n(E)}{l^- \Pi_{n+1}(E) + l^+ \Pi_{n-1}(E)} = -2l\pi_n(E) \ln 1 = 0\ , \end{aligned}$$

and vanishes only when $e^{2F}\Pi_{n+1}(E) = e^{-2F}\Pi_{n-1}(E)$. It is remarkable that the condition of vanishing entropy production $s_i(E) = 0$ is equivalent the condition of vanishing flow $J_{n|n+1}(E) = 0$.

The relation between the entropy production and the flow becomes more explicit when one considers the macroscopic limit. By setting $\Pi_n(E)/d = \pi(X,E)$, $F/d = \mathscr{F}$ and $l = \tau D/d^2$, and taking the limit of $\tau \to 0$ and $d \to 0$, we have

$$\begin{aligned} \sigma_i(E) &\equiv \frac{s_i(E)}{\tau d} \\ &= \frac{1}{D\pi(X,E)} \left(2D\mathscr{F}\pi(X,E) + D\frac{\partial \pi(X,E)}{\partial X} \right)^2\ , \end{aligned} \tag{69}$$

which is proportional to the square of the flow. Hence, the entropy production per unit time and per unit length is given by

$$\begin{aligned} \frac{\Delta_i S}{\tau d} &= \int_{E_0}^{\infty} \mathrm{d}E \sigma_i(E) \\ &= \int_{E_0}^{\infty} \mathrm{d}E \frac{1}{D\pi(X,E)} \left(2D\mathscr{F}\pi(X,E) + D\frac{\partial \pi(X,E)}{\partial X} \right)^2\ . \end{aligned} \tag{70}$$

In particular, when the energy distribution is independent of the spatial distribution

$$\pi(X,E) = p(X)h(E)\ ,\ \text{with}\ \int_{E_0}^{\infty} dE\, h(E) = 1\ ,$$

the entropy production $\Delta_i S$ becomes

$$\frac{\Delta_i S}{\tau d} = \frac{1}{Dp(X)} \left(2D\mathscr{F}p(X) + D\frac{\partial p(X)}{\partial X} \right)^2\ . \tag{71}$$

As discussed by Breymann et al. [20], this result agrees with classical thermodynamics since the entropy production is proportional to the square of the particle flow Eq. (58). We emphasize that the entropy production which is consistent with classical thermodynamics can be obtained for a conservative system. Therefore, in order to be consistent with classical thermodynamics, the inclusion of dissipation is not always necessary.

7 Conclusions

We have constructed a multibaker map with "kinetic energy" to which an external field can be applied. The map is volume-preserving, time-reversal symmetric and conserves total energy. In an appropriate macroscopic limit, the particle distribution obeys a Smoluchowski-type equation. For the cases without an external field and with a constant external force, the nonequilibrium stationary states are constructed by solving the evolution equation of the partially integrated distribution functions. These states are described by singular functions such as an incomplete Takagi function and Lebesgue's singular functions. Moreover, in an appropriate macroscopic limit, the mass flow for the stationary states is identical to the one expected from the Smoluchowski equation and there appears a "heat flow" proportional to the local energy gradient. The Gaspard–Gilbert–Dorfman entropy production is calculated for the stationary states and is shown to be positive. Particularly, for the case with a constant external force, when the energy distribution is independent of the spatial distribution, the macroscopic limit of the entropy production is consistent with classical thermodynamics. We give a few more remarks.

1. In contrast to the observation for the conventional multibaker map by Breymann et al. [20], the entropy production which is consistent with classical thermodynamics can be obtained for a conservative driven multibaker map. The key ingredient of our model is the variation of the phase-space volume with respect to the site coordinate n. This variation somehow plays the role of phase-space contraction and, as a result, the entropy production becomes similar to that for the dissipative multibaker model.

2. As discussed in the Appendix, there exists a close relation between the nonequilibrium stationary states of the present model and the dissipative multibaker map studied by Breymann, et al. [20] with "appropriate dissipation". Indeed, the stationary distributions for both cases are identical except for a trivial scaling factor.

3. The nonequilibrium stationary distributions derived in Sect. 5 are absolutely continuous with respect to the Lebesgue measure for the finite mutibaker chain. They become singular only for the infinite chain. As shown in the Appendix, this is also the case for a dissipative multibaker chain provided that the distributions of the particle reservoirs are uniform with respect to the Lebesgue measure. It is remarkable that the present conservative multibaker map shares common properties with the dissipative multibaker map.

4. Since the potential Φ is arbitrary and the evolution equation of the distribution function reduces to a Smoluchowski-type equation in the macroscopic limit, the present model for a random potential Φ provides a deterministic model of the Brownian motion in a random environment [26].

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Appendix: dissipative multibaker map

In this appendix, we construct nonequilibrium stationary states for the three-strip dissipative multibaker map studied by Breymann et al. [20].

$$B_{\rm d}(n,x,y) = \begin{cases} (n-1,\ \frac{x}{l^-},\ \tilde{l}^-\ y), & x \in [0, l^-), \\ (n,\ \frac{x-l^-}{s},\ sy+\tilde{l}^-), & x \in [l^-, 1-l^+), \\ (n+1,\ \frac{x-(1-l^+)}{l^+},\ \tilde{l}^+y+1-\tilde{l}^+), & x \in [1-l^+, 1], \end{cases} \tag{A1}$$

where $l^\pm > 0$, $\tilde{l}^\pm > 0, s > 0$ and $l^+ + l^- + s = \tilde{l}^+ + \tilde{l}^- + s = 1$. Note that, when $l^+ \neq \tilde{l}^+$ (or $l^- \neq \tilde{l}^-$), the map $B_{\rm d}$ is nonconservative. As before, the evolution equation for the partially integrated distribution function $G_t(n,x,y) \equiv \int_0^y dy' \rho_t(n,x,y')$ is derived where ρ_t is the density function at time t, and we look for its stationary solution under the boundary condition:

$$G_t(-1,x,y) = \rho_- y, \quad G_t(N+1,x,y) = \rho_+ y \ . \tag{A2}$$

Then we find that the partially integrated distribution function of the stationary state is independent of x and is given by

$$G_\infty(n,y) = A\left(\frac{l^+}{l^-}\right)^n g_n(y) + B h_n(y) \ , \tag{A3}$$

where the constants A and B depend on $\rho_\pm$ and the functions $g_n(y)$ and $h_n(y)$ are unique solutions of the functional equations

$$g_n(y) = \begin{cases} l^+\ g_{n+1}(y/\tilde{l}^-), & (0 \le y \le \tilde{l}^-), \\ s\ g_n[(y-\tilde{l}^-)/s] + l^+, & (\tilde{l}^- \le y \le 1-\tilde{l}^+), \\ l^-\ g_{n-1}[(y-1+\tilde{l}^+)/\tilde{l}^+] + 1 - l^-, & (1-\tilde{l}^+ \le y \le 1) \ , \end{cases} \tag{A4}$$

and

$$h_n(y) = \begin{cases} l^-\ h_{n+1}(y/\tilde{l}^-), & (0 \le y \le \tilde{l}^-), \\ s\ h_n[(y-\tilde{l}^-)/s] + l^-, & (\tilde{l}^- \le y \le 1-\tilde{l}^+), \\ l^+\ h_{n-1}[(y-1+\tilde{l}^+)/\tilde{l}^+] + 1 - l^+, & (1-\tilde{l}^+ \le y \le 1) \ , \end{cases} \tag{A5}$$

with the boundary conditions $g_{-1}(y) = g_{N+1}(y) = h_{-1}(y) = h_{N+1}(y) = y$. Note that, for a finite open multibaker chain, the functions g_n and h_n are differentiable with finite derivatives almost everywhere with respect to the Lebesgue measure and, as a result, the stationary measure is absolutely continuous with respect to the Lebesgue measure. The stationary measure is singular only for an infinite or closed multibaker chain.

The flow associated with the stationary state is

$$\begin{aligned} J_{n|n+1} &= l^+ G_\infty(n,1) - l^- G_\infty(n+1,1) \\ &= (l^+ - l^-)B \ , \end{aligned} \tag{A6}$$

which is independent of A. Because of this fact, we call the part of $G_\infty(n,y)$ proportional to B the flow-carrying component and the one proportional to A the flow-noncarrying component.

Breymann et al. [20] studied the entropy production of the nonconservative multibaker map Eq. (A1) and showed that the entropy production has a macroscopic limit consistent with classical thermodynamics if, and only if, the equalities $l^+ = \tilde{l}^-$ and $l^- = \tilde{l}^+$ hold. Now we show that their result can be restated such that the entropy production has a macroscopic limit consistent with classical thermodynamics if, and only if, the flow-noncarrying component for an infinitely long multibaker chain is absolutely continuous with respect to the Lebesgue measure. This observation seems to suggest a close relation between the entropy production and the singularity of the distribution.

The proof is easy. For an infinitely long multibaker chain, the function g_n reduces to the function g which is the solution of

$$g(y) = \begin{cases} l^+\ g(y/\tilde{l}^-), & 0 \le y \le \tilde{l}^-, \\ s\ g[(y-\tilde{l}^-)/s] + l^+, & \tilde{l}^- \le y \le 1-\tilde{l}^+, \\ l^-\ g[(y-1+\tilde{l}^+)/\tilde{l}^+] + 1 - l^-, & 1-\tilde{l}^+ \le y \le 1 \ , \end{cases} \tag{A7}$$

which is the cumulative function of the multinomial measure on the real axis. It is known [27] that the multinomial measure is absolutely continuous with respect to the Lebesgue measure if, and only if, the equalities $l^+ = \tilde{l}^-$ and $l^- = \tilde{l}^+$ hold. This completes the proof.

Finally we remark that there exists an interesting relation between our model and the Breymann–Tél–Vollmer model with "appropriate dissipation". Indeed, when the conditions $l^+ = \tilde{l}^-$ and $l^- = \tilde{l}^+$ are satisfied,

$g_n(y) = y$ and $h_n(y) = f_n(y)$, where $f_n(y)$ is the incomplete Lebesgue singular function Eq. (49) discussed in Sect. 5. Thus, we have

$$G_\infty(n, y) = A\left(\frac{l^+}{l^-}\right)^n y + Bf_n(y) \ , \tag{A8}$$

which, except for a scaling factor, is identical to the distribution Eq. (48) obtained in Sect. 5.2. (Note that in our model $l^+/l^- = \mathrm{e}^{-2F}$).

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